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THE CHI-SQUARED PROCESS WITH APPLICATIONS TO HYPOTHESIS TESTING AND TIME SERIES ANALYSIS

Robert B. Davies

May 1980

STATISTICAL LABORATORY

University of California

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where $\{X_1(\cdot), \dots, X_s(\cdot)\}$ is a multivariate Gaussian process such that for

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20. ABSTRACT (CONT.) (THETA)

each value of θ, the random variables $X_1(\theta), \ldots, X_s(\theta)$ are independent with zero means and unit variances. This enables a bound on the probability that the maximum of the process exceeds a fixed level to be obtained. The result is used to adapt the method of Davies (1977) to testing the hypothesis that a vector $\xi = 0$ when a nuisance parameter ∂ is present only under the alternative $\xi \neq 0$. The method is then applied to the problem of detecting a discrete frequency component in a Gaussian time series.

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THE CHI-SQUARED PROCESS WITH APPLICATIONS TO HYPOTHESIS TESTING AND TIME SERIES ANALYSIS

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ABSTRACT

We calculate the expected number of upcrossings of a fixed level by the process

$$S(\theta) = X_1^2(\theta) + \dots + X_s^2(\theta)$$

where $\{X_1(\cdot), \dots, X_s(\cdot)\}$ is a multivariate Gaussian process such that for each value of θ, the random variables $X_1(\theta), \dots, X_s(\theta)$ are independent with zero means and unit variances. This enables a bound on the probability that the maximum of the process exceeds a fixed level to be obtained. The result is used to adapt the method of Davies (1977) to testing the hypothesis that a vector $\xi = 0$ when a nuisance parameter θ is present only under the alternative $\xi \neq 0$. The method is then applied to the problem of detecting a discrete frequency component in a Gaussian time series.

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1. INTRODUCTION

Davies (1977) considered the problem of testing the hypothesis that a single parameter $\xi = 0$ against the alternative $\xi > 0$ when a single nuisance parameter θ was present only under the alternative. In this paper we consider the problem of testing the hypothesis that all components of an s-dimensional vector ξ are zero against the alternative that at least one of them is non-zero, where, again, there is a nuisance parameter, θ, which enters into the model only when ξ is non-zero. As in Davies (1977) standard asymptotic methods cannot be applied directly. However, by using them or some other method, it may be possible, for each θ, to find a random variable $S(\theta)$ which has, at least asymptotically, a chi-squared distribution when $\xi = 0$ and which would be suitable for testing $\xi = 0$ against $\xi \neq 0$ if θ were known. Then the test we consider in this paper is to reject the hypothesis for large values of

$$\sup \{S(\theta) : L \leq \theta \leq U\} \tag{1.1}$$

where $[L,U]$ is the range of possible values for θ. The problem is to find the distribution of (1.1) so that significance probabilities can be calculated.

To simplify the discussion we suppose that $\xi = 0$ and sample sizes are so large that deviations from an exact chi-squared distribution can be ignored. Suppose that $S(\theta)$ can be represented

$$S(\theta) = X_1^2(\theta) + \ldots + X_s^2(\theta) \tag{1.2}$$

where the $X_j(\theta)$ form a multivariate Gaussian process such that for each θ the $X_j(\theta)$ are independent with standard normal distributions. Then we will say that $S(\theta)$ is a chi-squared process. This is a generalization of the process considered by Sharpe (1978) since we do not require $X_i(\theta_1)$ and $X_j(\theta_2)$ to be independent if $\theta_1 \neq \theta_2$ nor do we require stationarity.

As in Davies (1977) we find a bound on the probability that $\sup \{S(\theta)\}$ exceeds a given value by finding the expected number of upcrossings of a fixed level by the process $S(\theta)$. This is done in section 2; in section 3 the result is applied to the testing problem.

An important example of the test is concerned with the detection of frequency components in a time-series. Suppose $Z_1, \ldots, Z_n$ is a sequence of independent normal random variables with unit variances and

$$EZ_j = \xi_1 \sin(j\theta) + \xi_2 \cos(j\theta) \tag{1.3}$$

with $0 \leq U \leq \theta \leq L \leq \pi$. That is, if $\xi \neq 0$, a discrete frequency component is present in the series. This example is considered in section 4.

We use the following notation: A^* denotes the transpose of a vector or matrix, A; 1 denotes an indicator function.

2. UPCROSSINGS OF A CHI-SQUARED PROCESS

We suppose that $X(\theta):\theta\in[L,U]$ is an s-dimensional Gaussian process, with zero expectation and a continuous derivative $Y(\theta) = dX(\theta)/d\theta$. We further suppose that for each θ the components of $X(\theta)$ are independent with unit variance. Then $X(\theta)$ and $Y(\theta)$ are jointly normally distributed, $Y(\theta)$ also having zero expectation. Suppose

$$\mathrm{Var}\begin{pmatrix} X(\theta) \\ Y(\theta) \end{pmatrix} = \begin{pmatrix} I & A(\theta) \\ A^*(\theta) & B(\theta) \end{pmatrix} \tag{2.1}$$

Then $A(\theta)$ is skew-symmetric since

$$\begin{aligned} A(\theta) + A^*(\theta) &= \lim_{\Delta\to 0} E[X(\theta)\{X(\theta + \Delta) - X(\theta)\}^* \\ &\quad + \{X(\theta + \Delta) - X(\theta)\}X^*(\theta + \Delta)] \;/\; \Delta \\ &= 0. \end{aligned}$$

Let $S(\theta) = X^*(\theta)X(\theta)$ as in (1.2),

$$T(\theta) = dS(\theta)/d\theta = 2X^*(\theta)Y(\theta) \tag{2.2}$$

and let $f(\cdot)$ denote the density of $S(\theta)$, a chi-squared variable with s degrees of freedom.

Then for $u > 0$ one can check that the conditions of Marcus (1977), section 5, are satisfied. Following Sharpe (1978), the expected number of upcrossings of the level u is given by

$$\int_L^U \psi(\theta)d\theta \tag{2.3}$$

where $\psi(\theta) = E(T(\theta)1_{T(\theta)>0} \mid S(\theta) = u)f(u).$ (2.4)

In (2.2) $Y(\theta)$ can be replaced by $Y(\theta) - A^*(\theta)X(\theta)$ in which case (2.1) must be replaced by

$$\mathrm{Var}\begin{pmatrix} X(\theta) \\ Y(\theta) - A^*(\theta)X(\theta) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B(\theta) - A^*(\theta)A(\theta) \end{pmatrix} \tag{2.5}$$

Now choose $U(\theta)$ orthogonal so that

$$U(\theta)\{B(\theta) - A^*(\theta)A(\theta)\}U^*(\theta) = \Lambda(\theta) \tag{2.6}$$

where $\Lambda(\theta)$ is diagonal. Premultiplying $X(\theta)$, $Y(\theta)$ by $U(\theta)$ will not affect (1.2) and (2.2) and so, in these equations, we can assume that the components of $X(\theta)$, $Y(\theta)$ are independent, those of $X(\theta)$ having unit variance and those of $Y(\theta)$ having variances given by the elements of $\Lambda(\theta)$, the eigenvalues of $B(\theta) - A^*(\theta)A(\theta)$.

Before proceeding with the evaluation of (2.4) we note how $\Lambda(\theta)$ may be found from the covariance function of $X(\theta)$. Let

$$R_{\theta\Delta} = E\{X(\theta + \Delta)X^*(\theta)\} \tag{2.7}$$

Then

$$R_{\theta\Delta} - I = E[\{X(\theta + \Delta) - X(\theta)\}X^*(\theta)] = \Delta A^*(\theta) + o(\Delta)$$

$$2I - R_{\theta\Delta} - R_{\theta\Delta}^* = E[\{X(\theta + \Delta) - X(\theta)\}\{X(\theta + \Delta) - X(\theta)\}^*] = \Delta^2 B(\theta) + o(\Delta^2)$$

Now

$$R_{\theta\Delta}\exp\{-\tfrac{1}{2}(R_{\theta\Delta}-R_{\theta\Delta}^{*})\}$$

$$=\{I-\tfrac{1}{2}(R_{\theta\Delta}-R_{\theta\Delta}^{*})+\tfrac{1}{2}(R_{\theta\Delta}+R_{\theta\Delta}^{*}-2I)\}$$

$$\cdot\{I-\tfrac{1}{2}(R_{\theta\Delta}-R_{\theta\Delta}^{*})+\frac{1}{8}(R_{\theta\Delta}-R_{\theta\Delta}^{*})^{2}\}+o(\Delta^{2})$$

$$=I-\tfrac{1}{2}\Delta^{2}\{B(\theta)-A^{*}(\theta)A(\theta)\}+o(\Delta^{2})$$

But $\exp\{-\tfrac{1}{2}(R_{\theta\Delta}-R_{\theta\Delta}^{*})\}$ is orthogonal. Hence a singular value decomposition of $R_{\theta\Delta}$ can be expressed as

$$I-\tfrac{1}{2}\Delta^{2}\Lambda(\theta)+o(\Delta^{2}) \tag{2.8}$$

This provides the alternative way of finding $\Lambda(\theta)$.

Returning to (2.4), dropping references to θ and supposing

$$\operatorname{Var}\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}I & 0\\0 & \Lambda\end{pmatrix} \tag{2.9}$$

we have

$$\psi=E(T1_{T>0}|S=u)f(u)$$

$$=2E\{E(X^{*}Y1_{X^{*}Y>0}|X^{*}X,Y)|X^{*}X=u\}f(u)$$

Now

$$E(X^{*}Y1_{X^{*}Y>0}|X^{*}X,Y)=c_{Y}(X^{*}X)^{\frac{1}{2}}$$

for some c_Y which may depend on Y but not on X^*X since X and Y are independent and the conditional distribution of X given X^*X is just

uniform on the surface of the sphere of radius $(X^*X)^{\frac{1}{2}}$. To evaluate c_Y take expectations:

$$E(X^*Y 1_{X^*Y>0} | Y) = c_Y E(X^*X)^{\frac{1}{2}}$$

On the left hand side, rotate X so that X_1 lies in the direction of Y to obtain

$$E(X_1 1_{X_1>0}) \cdot (Y^*Y)^{\frac{1}{2}} = (2\pi)^{-\frac{1}{2}} (Y^*Y)^{\frac{1}{2}}$$

Combining these results and substituting

$$f(u) = u^{s/2-1} e^{-u/2} \ / \ \{\Gamma(s/2) 2^{s/2}\}$$

$$E(X^*X)^{\frac{1}{2}} = 2^{\frac{1}{2}} \Gamma\{(s+1)/2\} \ / \ \Gamma(s/2)$$

we find

$$\psi(\theta) = E\{Y^*(\theta)Y(\theta)\}^{\frac{1}{2}} \frac{u^{(s-1)/2} e^{-u/2}}{\pi^{\frac{1}{2}} 2^{s/2} \Gamma\{(s+1)/2\}} \qquad (2.10)$$

where $Y(\theta)$ is a vector of independent centered normal random variables with variances given by the elements of $\Lambda(\theta)$.

Hence we have

Theorem 2.1 If the s-dimensional Gaussian process $X(\theta)$ has continuously differentiable sample paths for $L \leqslant \theta \leqslant U$ and $E\{X(\theta)\} = 0$, $\mathrm{Var}\{X(\theta)\} = I$ for each θ then the expected number of upcrossings of the level u in the range $\theta \in [L,U]$ is given by (2.3) where $\psi(\theta)$ is defined by (2.10) and $Y(\theta)$

is composed of s independent centered normal random variables with variances $\lambda_1(\theta), \dots \lambda_s(\theta)$ being the eigenvalues of $B(\theta) - A^*(\theta)\, A(\theta)$; $A(\theta)$ and $B(\theta)$ being as in (2.1).

Corollary 2.2.

$$\mathrm{pr}[\sup\{X(\theta): L \leqslant \theta \leqslant U\} > u] \leqslant \int_L^U \psi(\theta)d\theta + \mathrm{pr}(\chi_s^2 > u) \tag{2.11}$$

where χ_s^2 denotes a chi-squared random variable with s degrees of freedom.

The corollary is proved in the same way as formula (3.6) in Davies (1977). Sharpe (1978) has shown that the number of high level upcrossings in the stationary independent case is approximately Poisson and we would expect this to be true more generally. Hence, as in Davies (1977), we would expect the bound (2.11) to be reasonably sharp.

It remains to consider the calculation of $E\{(Y^*Y)^{\frac{1}{2}}\}$. Harvey (1965) gives the general formula

$$E\{(Y^*Y)^{\frac{1}{2}}\} = \frac{1}{2\pi^{\frac{1}{2}}} \int_0^\infty \frac{\rho(t) + \sin F(t) - \cos F(t)}{t^{3/2}\, \rho(t)}\, dt \tag{2.12}$$

where

$$F(t) = \tfrac{1}{2} \sum_1^s \arctan(\lambda_i t)$$

$$\rho(t) = \prod_1^s (1 + \lambda_i^2 t^2)^{\frac{1}{4}}$$

Formula (2.12) can also be expressed

$$E\{(Y^*Y)^{\frac{1}{2}}\} = (2\pi)^{-\frac{1}{2}} \int_0^\infty \{1 - \prod_{j=1}^s (1 + \lambda_j t)^{-\frac{1}{2}}\} t^{-3/2} dt.$$

Alternatively, Harvey (1965) gives a variety of approximate expressions.

We consider two special cases. Suppose all the λ_i are equal with common value λ. Then

$$E\{(Y^*Y)^{\frac{1}{2}}\} = (2\lambda)^{\frac{1}{2}}\, \Gamma\{(s+1)/2\} \,/\, \Gamma(s/2)$$

leading to a result in agreement with Sharpe's (1978) formula (3.2). If $s = 2$ then $E\{(Y^*Y)^{\frac{1}{2}}\}$ can be evaluated directly (Harvey, 1965) as

$$\begin{aligned} E\{(Y^*Y)^{\frac{1}{2}}\} &= (2\lambda_1/\pi)^{\frac{1}{2}}\, E(1 - \lambda_2/\lambda_1) \text{ if } \lambda_1 \geqslant \lambda_2 \\ &= (2\lambda_2/\pi)^{\frac{1}{2}}\, E(1 - \lambda_1/\lambda_2) \text{ if } \lambda_2 \geqslant \lambda_1 \qquad (2.13) \end{aligned}$$

where, on the right hand side, E denotes a complete elliptic integral of the second kind; see Abramowitz and Stegun (1970), formula 17.3.3.

3. HYPOTHESIS TESTING

Suppose that the outcome of an experiment is represented by a (vector) random variable Z, the distribution of which depends on parameters $\xi_1, \ldots, \xi_s$ and θ where θ is known to lie in the closed interval $[L,U]$, and that we wish to test the hypothesis that all the ξ_j are zero against the alternative that at least one of them is non-zero. Now suppose that the distribution of Z does not depend on θ when $\xi = (\xi_1, \ldots, \xi_s)^* = 0$ so that, for example, standard asymptotic methods cannot be applied directly. However, if θ was known, it might be possible to find a test which rejected the hypothesis $\xi = 0$ for large values of the statistic

$$S(\theta) = X_1^2(\theta) + \ldots + X_s^2(\theta) \qquad (3.1)$$

where the $X_j(\theta)$ were, for each θ, at least approximately independently normally distributed with unit variances and zero means when $\xi = 0$. For example, if $\hat{\xi}(\theta)$ represented the maximum likelihood estimator for ξ when θ was given and $\Sigma(\theta)$ its asymptotic variance matrix one might take

$$X(\theta) = \Sigma^{-\frac{1}{2}}(\theta)\hat{\xi}(\theta).$$

Alternatively one might derive a similar formula from formula (55) of Bühler and Puri (1966).

The test suggested here is to reject the hypothesis $\xi = 0$ if

$$\sup\{S(\theta): L \leqslant \theta \leqslant U\} > u. \qquad (3.2)$$

Formula (2.11) can then be used to give an upper bound on the probability of type I error or on the significance probability.

One can also find a lower bound on the power using the formula

$$\mathrm{pr}_{\xi,\theta_0}[\{\sup S(\theta): L \leqslant \theta \leqslant U\} > u] \leqslant \mathrm{pr}_{\xi,\theta_0}\{S(\theta_0) > u\} \qquad (3.3)$$

which we would be able to evaluate using a non-central chi-squared distribution. The analogous bound of Davies (1977) was found to provide an adequate approximation; however no studies have been carried out in the present instance.

We now consider a particular non-asymptotic situation in which it is possible to simplify the finding of the eigenvalues required by Theorem 2.1.

Suppose we observe $Z = (Z_1, \ldots, Z_n)^*$ where the Z_1 are independently normally distributed with unit variances and with

$$EZ = W^*(\theta)\xi \qquad (3.4)$$

where $W(\theta)$ is an $s \times n$ dimensional matrix of rank s. Then if θ were known the most stringent test for testing $\xi = 0$ against $\xi \neq 0$ rejects the hypothesis for large values of

$$S(\theta) = Z^*Q(\theta)Z \qquad (3.5)$$

where

$$Q(\theta) = W^*(\theta)\{W(\theta)W^*(\theta)\}^{-1}W(\theta).$$

Factorize

$$Q(\theta) = U^*(\theta)U(\theta) \tag{3.6}$$

where $U(\theta)$ is an $s \times n$ matrix with $U(\theta)U^*(\theta) = I$. Then letting $X(\theta) = U(\theta)Z$ puts the problem in the form already discussed. Then $Y = \{dU(\theta)/d\theta\}Z$ and so

$$A(\theta) = U(\theta)\{dU^*(\theta)/d\theta\}$$

$$B(\theta) = \{dU(\theta)/d\theta\}\{dU^*(\theta)/d\theta\}.$$

It will be convenient to write $F \equiv G$ if matrices F and G have the same non-zero eigenvalues. In particular if the products FG and GF are defined $FG \equiv GF$. We need to find the eigenvalues of

$$B - A^*A = B + A^2 + (A^*)^2 + AA^*$$

$$\equiv U^*\left\{\frac{dU}{d\theta}\frac{dU^*}{d\theta} + U\frac{dU^*}{d\theta}U\frac{dU^*}{d\theta} + \frac{dU}{d\theta}U^*\frac{dU}{d\theta}U^* + U\frac{dU^*}{d\theta}\frac{dU}{d\theta}U^*\right\}U$$

$$= Q\left\{\frac{dQ}{d\theta}\right\}^2 Q. \tag{3.7}$$

If we write

$$R(\theta) = W^*(\theta)\{W(\theta)W^*(\theta)\}^{-1}(dW(\theta)/d\theta)$$

one can check that

$$Q\left\{\frac{dQ}{d\theta}\right\}^2 Q = R(I - Q)R \tag{3.8}$$

$$= W^*(WW^*)^{-1}\frac{dW}{d\theta}\{I - W^*(WW^*)^{-1}W\}\frac{dW^*}{d\theta}(WW^*)^{-1}W$$

$$\equiv (WW^*)^{-\frac{1}{2}}\{\frac{dW}{d\theta}\frac{dW^*}{d\theta} - \frac{dW}{d\theta}W^*(WW^*)^{-1}W\frac{dW^*}{d\theta}\}(WW^*)^{-\frac{1}{2}} \tag{3.9}$$

Thus the non-zero elements of $\Lambda(\theta)$ can be found as the non-zero eigen-values of (3.7), (3.8) or (3.9). Formula (3.9) is particularly convenient if

$$WW^*,\ (dW/d\theta)(dW^*/d\theta) \text{ and } (dW/d\theta)W^* \tag{3.10}$$

are all diagonal because we can set

$$\Lambda = (WW^*)^{-1}\left(\frac{dW}{d\theta}\frac{dW^*}{d\theta}\right) - (WW^*)^{-2}\left(\frac{dW}{d\theta}W^*\right)^2. \tag{3.11}$$

4. DETECTION OF A DISCRETE FREQUENCY COMPONENT

We observe $Z = (Z_1, \ldots , Z_n)^*$ where the Z_j are independently normally distributed with unit variances and with

$$EZ_j = \xi_1 \sin[\{j - (n+1)/2\}\theta] + \xi_2 \cos[\{j - (n+1)/2\}\theta]. \qquad (4.1)$$

That is $Z_1, \ldots , Z_n$ is a sequence of independent standard normal variables on to which has been superimposed a cyclic effect with period $2\pi/\theta$. Formula (4.1) is just a change of parameterization of (1.3). Now suppose we wish to test the hypothesis, $\xi = 0$, that is there is no frequency component, against the alternative that $\xi \neq 0$. Traditionally, see Hannan (1960), pages 76-83, this problem has been handled by looking at only values of θ of the form $2\pi k/n : k = 1, \ldots [n/2]$. For these values of θ the corresponding values of (3.5) will be independent and so significance levels can be readily calculated. However a loss of power occurs if the true value of θ falls between these values. We should emphasize that we are concerned with discrete frequency components. The method considered here has little relevance to the problem of detecting peaks in the frequency spectrum which have a bandwidth greater than $2\pi/n$ cycles per sampling interval.

Now apply the theory of the previous section. The matrices (3.10) are derived in the appendix and all turn out to be diagonal so (3.11) applies. Applying (3.5) and (4.1) we find

$$S(\theta) = \left\{\sum_1^n Z_j \sin[\{j - (n+1)/2\}\theta]\right\}^2 / v_1$$

$$+ \left\{\sum_1^n Z_j \cos[\{j - (n+1)/2\}\theta]\right\}^2 / v_2 \qquad (4.2)$$

where

$$v_1 = \{n - \sin(n\theta) / \sin(\theta)\} / 2$$

$$v_2 = \{n + \sin(n\theta) / \sin(\theta)\} / 2$$

For the moment we suppose $0 < L \leqslant \theta \leqslant U < \pi$ so that $S(\theta)$ is defined. Then it is shown in the appendix that the eigenvalues λ_1 and λ_2 can be expressed

$$(n^2 - 1)/(3G) - n^2/4 + (1 - F^2/G^2)/(4 \sin^2 \theta) \qquad (4.3)$$

where

$$F = \cos(n\theta) - \sigma \cos \theta$$

$$G = 1 - \sigma \sin(n\theta)/(n \sin \theta)$$

$\sigma = +1$ to give λ_1 , -1 to give λ_2.

For each θ the value of $E\{(Y^*Y)^{\frac{1}{2}}\} = a(\theta)$, say, can be found from (2.13) and so the bound (2.11) is equal to

$$\int_L^U a(\theta)\, d\theta \,.\, u^{\frac{1}{2}} e^{-u/2} / \pi + e^{-u/2}. \qquad (4.4)$$

In fact (4.3) tends to zero as θ tends to 0 or π so, provided (4.2) and (4.3) are defined by continuity at $\theta = 0$ or π, we can allow θ to take any value in the range $[0,\pi]$. For n large and θ not near 0 or π formula (4.3) can be approximated by $n^2/12$ leading to $a(\theta)$ in (4.4) being approximated by $n(\pi/24)^{\frac{1}{2}}$ which in turn leads to (4.4) being approximated by

$$n\, u^{\frac{1}{2}}\, e^{-u/2}\, (U-L) \;/\; (24\pi)^{\frac{1}{2}} + e^{-u/2} \tag{4.5}$$

The approximation of $a(\theta)$ by $n(\pi/24)^{\frac{1}{2}}$ turns out to be adequate if $n > 4$ and $2\pi/n < \theta < \pi - 2\pi/n$. This approximation is also good for moderate and large values of n when θ covers the whole range $0 \leqslant \theta \leqslant \pi$. In Table I the values of $\int_0^\pi a(\theta)\, d\theta \,/\, \pi$ and its approximation $n(\pi/24)^{\frac{1}{2}}$ are listed for various values of n. It will be noted that the approximation is especially good if the approximate value is reduced by .5.

To test the sharpness of the bound (4.4) on the significance probability a simulation was carried out with $n = 16$. One thousand simulations were performed. The function $S(\theta)$ was examined at 255 points and the number of times the hypothesis, $\xi = 0$, was rejected at the .2, .1, .05, .02 and .01 levels counted together with the total number of upcrossings. The results are listed in Table II. In each case the number of upcrossings was only slightly above the number of significant results indicating that the nominal significance level was close to the actual significance level. Other simulations with $n = 4$ and $n = 64$ yielded similar results. The simulations were carried out on the CDC 6400 computer on the Berkeley campus of the University of California and used the IMSL subroutines GGNML and FFTRC.

Formula (4.5) can be compared with the formula one obtains when one considers the test based on the maximum of $S_k = S(2\pi k/n)$ where k is an integer. Assuming U and L to be of the form $(2k+1)\pi/n$

$$P\{\sup(S_k : L \leqslant 2\pi k/n \leqslant U) > u\}$$

$$= 1 - (1 - e^{-u/2})^{n(U-L)/(2\pi)}$$

$$\sim ne^{-u/2}\,(U-L)\,/\,(2\pi). \tag{4.6}$$

In fact, there is relatively little difference between the values of u required to give the same value to formulas (4.5) and (4.6) and thus there is only a small loss of sensitivity when the test based on the maximum of the S_k is replaced by the test considered in this paper and θ is of the form $2\pi k/n$. On the other hand for values of θ of the form $(2k+1)\pi/n$ our test has a substantial advantage.

In practice, of course, one would need to normalize the time-series $(Z_1, \ldots Z_n)$ by subtracting off the sample mean and dividing by the sample standard deviation. It may also be necessary to compensate for serial correlation by fitting a simple autoregressive or moving average process. Subtracting the sample mean would have a major effect only at the very low frequency end of the spectrum and hence for large n formula (4.5) would still be applicable. On the other hand if we denote (4.5) by $\alpha(u)$ and differentiate we find $d\alpha(u)/du = O(\alpha(u))$. Also $u = O(\log n)$. Since the standard deviation and any autoregressive or moving average parameters would be estimated to an accuracy of $O(n^{-\frac{1}{2}})$ the effect on the significance level

would be similar to that of changing u by $O(un^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}} \log n)$ which would be asymptotically negligible. Hence, for large n, formula (4.5) would still be applicable when applied to the suitably pre-whitened normalized, and centered time-series.

According to Feller (1970), section 26.7, theorems 1 and 3, the central limit theorem holds for moderate deviations of $o(n^{1/6})$ and hence we would expect (4.5) to hold asymptotically, even if the Z_j were not normally distributed. However further study is required on this point.

TABLE I

n	$\int_0^\pi a(\theta)\, d\theta\, /\pi$	$n(\pi/24)^{\frac{1}{2}}$
5	1.26	1.81
10	3.09	3.62
15	4.91	5.43
20	6.72	7.23
25	8.53	9.05
30	10.34	10.85
40	13.96	14.47
50	17.58	18.09
60	21.20	21.71
80	28.44	28.94
100	35.68	36.18

TABLE II

Simulation of the Test for Frequency Components

Nominal Significance Level	.2	.1	.05	.02	.01
Critical level, u	8.848	10.385	11.901	13.879	15.362
Expected number of upcrossings	200	100	50	20	10
Observed number of upcrossings	221	114	57	21	10
Number of significance tests	198	107	56	20	9

APPENDIX

Derivation of Formula (4.3)

Using the notation of section 3 and letting $m = (n+1)/2$

$$W = \begin{bmatrix} \sin\{(1-m)\theta\} & \sin\{(2-m)\theta\} & \ldots & \sin\{(n-m)\theta\} \\ \cos\{(1-m)\theta\} & \cos\{(2-m)\theta\} & \ldots & \cos\{(n-m)\theta\} \end{bmatrix}.$$

All the off-diagonal elements of the matrices (3.10) are zero since the k-th and $(n+1-k)$-th terms in the sums that form these elements cancel. The diagonal elements of WW^* may be expressed

$$\sum_1^n \sin^2\{(k-m)\theta\} \quad \text{and} \quad \sum_1^n \cos^2\{(k-m)\theta\} \tag{A.1}$$

which are equal to $\frac{1}{2}nG$ where G is as in (4.3) with $\sigma = +1$ for the first term and -1 for the second term. The diagonal elements of $(dW/d\theta)W^*$ are given by

$$\sum_1^n (k-m) \sin\{(k-m)\theta\} \cos\{(k-m)\theta\} \quad \text{and} \quad -\sum_1^n (k-m)\{\sin\{(k-m)\theta\} \cos\{(k-m)\theta\} \tag{A.2}$$

and may be evaluated by differentiating (A.1) and dividing by 2. Hence they are equal to $-n(G \cos\theta + \sigma F) / (4 \sin\theta)$ with F and G as in (4.3) and σ as before. The diagonal elements of $(dW/d\theta)(dW^*/d\theta)$ are given by

$$\sum_1^n (k-m)^2 \cos^2\{(k-m)\theta\} \quad \text{and} \quad \sum_1^n (k-m)^2 \sin^2\{(k-m)\theta\} \tag{A.3}$$

Summing these two terms we obtain

$$\sum_{1}^{n} (k-m)^2 = n(n^2-1) / 12$$

while the difference may be found by differentiating (A.2):

$$\sum_{1}^{n} (k-m)^2[\cos^2\{(k-m)\theta\} - \sin^2\{(k-m)\theta\}]$$

$$= \tfrac{1}{4}\,\sigma n\{(n^2-1) + (G + G\cos^2\theta + 2\sigma F\,\cos\theta - n^2 G\sin^2\theta) / \sin^2\theta\}$$

for $\sigma = +1$ or -1. Thus (A.3) is equal to

$$n(n^2-1)/6 + n(G + G\cos^2\theta + 2\sigma F\,\cos\theta - n^2 G\sin^2\theta) / (8\sin^2\theta)$$

with $\sigma = +1$ *for the first term and* -1 *for the second term.* This completes the derivation of the matrices (3.10). Evaluation of (3.11) results in formula (4.3).

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